

Nonlocal filtration equations with rough kernels

by
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Dedicated to Juan Luis Vázquez, who has generously shared with us his deep insight on the subject of nonlinear diffusion, on the occasion of his 70th birthday

Abstract

We study the nonlinear and nonlocal Cauchy problem

$$\partial_t u + \mathcal{L}\varphi(u) = 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \quad u(\cdot, 0) = u_0,$$

where \mathcal{L} is a Lévy-type nonlocal operator with a kernel having a singularity at the origin as that of the fractional Laplacian. The nonlinearity φ is nondecreasing and continuous, and the initial datum u_0 is assumed to be in $L^1(\mathbb{R}^N)$. We prove existence and uniqueness of weak solutions. For a wide class of nonlinearities, including the porous media case, $\varphi(u) = |u|^{m-1}u$, $m > 1$, these solutions turn out to be bounded and Hölder continuous for $t > 0$. We also describe the large time behaviour when the nonlinearity resembles a power for $u \approx 0$ and the kernel associated to \mathcal{L} is close at infinity to that of the fractional Laplacian.

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1 Introduction and main results

We study the nonlinear and nonlocal Cauchy problem

$$(P) \quad \begin{cases} \partial_t u + \mathcal{L}\varphi(u) = 0, & (x, t) \in Q := \mathbb{R}^N \times \mathbb{R}_+, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^N, \end{cases}$$

with initial data $u_0 \in L^1(\mathbb{R}^N)$. Sign changes are allowed. The nonlocal operator \mathcal{L} is defined formally by

$$(1.1) \quad \mathcal{L}f(x) = \text{P.V.} \int_{\mathbb{R}^N} (f(x) - f(y)) J(x, y) dy,$$

with a measurable kernel J which is assumed to satisfy

$$(H_J) \quad \begin{cases} J(x, y) \geq 0, & J(x, y) = J(y, x), \\ \frac{\mathbf{1}_{\{|x-y| \leq 3\}}}{\Lambda |x-y|^{N+\sigma}} \leq J(x, y) \leq \frac{\Lambda}{|x-y|^{N+\sigma}}, & x, y \in \mathbb{R}^N, x \neq y, \end{cases}$$

for some constants $\sigma \in (0, 2)$ and $\Lambda > 0$. When $J(x, y) = |x - y|^{-(N+\sigma)}$, \mathcal{L} is a multiple of the fractional Laplacian $(-\Delta)^{\sigma/2}$, whose action on smooth functions is well defined and has a pointwise meaning. However, the pointwise expression (1.1) may not have sense for more general kernels in the class that we are considering here, even if f is very smooth. Hence we have to deal with weak solutions to give sense both to the time derivative and to the nonlocal operator. The precise definition of a weak solution, in terms of a bilinear form associated to the kernel J , is given in Section 2, which is devoted to some preliminaries.

The upper bound in (H_J) implies in particular that the operator is of Lévy type, $\int_{\mathbb{R}^N} \min(1, |x-y|^2) J(x, y) dy < \infty$ for almost every $x \in \mathbb{R}^N$. Moreover, the singularity on the diagonal $x = y$ is that of the fractional Laplacian. Thus, \mathcal{L} can be seen as an integro-differential operator of order σ with bounded measurable coefficients. The bounds in (H_J) allow the kernels J to be very oscillating and irregular. That is why they are referred to as *rough* kernels. Observe that rapidly decreasing or even compactly supported kernels are permitted. Once they are of Lévy type, what matters in what follows is their singularity at the origin.

The linear operator $\partial_t + \mathcal{L}$ is often described in the literature as a nonlocal diffusion operator, since on the one hand it is clear from the integral representation of \mathcal{L} that it is nonlocal, and on the other hand it can be regarded as a diffusion operator, in the sense that solutions to $\partial_t u + \mathcal{L}u = 0$ try to avoid high concentrations. The same is true for our nonlinear operator $\partial_t + \mathcal{L}\varphi(\cdot)$.

The nonlinearity φ is continuous and nondecreasing, and may be assumed without loss of generality to satisfy $\varphi(0) = 0$. The local analogue $\partial_t u - \Delta\varphi(u) = 0$ is known as the filtration equation. That is the reason why, by analogy, we label our equation as a *nonlocal filtration equation*. The typical example is that of powers, $\varphi(s) = |s|^{m-1}s$,

which includes both the case of nonlocal porous media, $m > 1$, and nonlocal fast diffusion, $0 < m < 1$. But we consider also more general functions.

To begin with, in Section 3 we prove existence, uniqueness, and a couple of important properties for bounded weak solutions.

Theorem 1.1 *Let J satisfy (H_J) , φ be continuous and nondecreasing, and $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.*

(a) *There exists a unique bounded weak solution to the Cauchy problem (P). It satisfies $\|u(\cdot, t)\|_\infty \leq \|u_0\|_\infty$ for every $t > 0$.*

(b) *If u and v are solutions to problem (P), they satisfy the T -contraction property*

$$\int_{\mathbb{R}^N} (u(\cdot, t) - v(\cdot, t))_+ \leq \int_{\mathbb{R}^N} (u(\cdot, 0) - v(\cdot, 0))_+ \quad \text{for all } t \geq 0.$$

(c) *If moreover $|\varphi(u)| \leq C|u|^m$, $m > \frac{(N-\sigma)_+}{N}$, and $J(x, y) = \tilde{J}(x - y)$, then $\int_{\mathbb{R}^N} u(\cdot, t) = \int_{\mathbb{R}^N} u_0$ for all $t \geq 0$.*

Remarks. (i) Existence and uniqueness of bounded *distributional* solutions to (P) have been recently obtained in [27] for more general operators \mathcal{L} than the ones considered here. In the present paper we sacrifice such generality in order to obtain stronger results.

(ii) We conjecture that conservation of mass is also true in the limit case $m = \frac{N-\sigma}{N}$ if $N > \sigma$, as it holds when \mathcal{L} is the fractional Laplacian; see [24].

(iii) Conservation of mass holds for kernels satisfying (H_J) more general than those considered in paragraph (c), as long as expression (1.1) is well defined for smooth functions; see the beginning of Section 2 for some conditions, either on σ or on J , guaranteeing this fact.

We next prove the continuity of bounded solutions when the nonlinearity satisfies

$$(H_\varphi) \quad \varphi \in C^1(\mathbb{R}), \quad \varphi(0) = 0, \quad \varphi'(s) > 0 \quad \text{for } s \neq 0.$$

Notice that φ can be degenerate at the level 0. However, we are leaving out nonlinearities which are too degenerate, like the Stefan one, $\varphi(s) = (s - 1)_+$, or singular, like the one corresponding to fast diffusion. If, moreover,

$$(H'_\varphi) \quad \begin{aligned} C_1 \frac{\varphi(r)}{r} &\leq \varphi'(s) \leq C_2 \frac{\varphi(r)}{r} \quad \text{for } 0 < |r| < 1, \quad |s| \in (|r|/4, 3|r|), \\ \sup_{[A, B]} \varphi' &\leq D_M \frac{\varphi(B) - \varphi(A)}{B - A}, \quad \text{if } -M \leq A < B \leq M, \end{aligned}$$

for some constants $0 < C_1 < C_2$ and $D_M > 0$, we get Hölder regularity. These conditions control the oscillation of the nonlinearity close to the origin, and are only

needed to deal with points at which the equation is degenerate. They are satisfied for example if

$$c_1|r|^{m-1} \leq \varphi'(r) \leq c_2|r|^{m-1} \quad \text{for } 0 \leq |r| \leq 1,$$

for some constants $0 < c_1 \leq c_2$, $m \geq 1$.

Theorem 1.2 *Let J and φ satisfy respectively (H_J) and (H_φ) , and let u be a bounded weak solution to the Cauchy problem (P). Then u is continuous in Q . If φ satisfies in addition (H'_φ) , then, for all $\tau > 0$ there is some $\alpha \in (0, 1)$ such that $u \in C^\alpha(\mathbb{R}^N \times (\tau, \infty))$.*

In the proof, performed in Section 4, we will use De Giorgi's method; see [22]. Thus, we will prove that the oscillation of the solution in space-time σ -cylinders of radius R ,

$$\mathcal{C}_R = \{|x - x_0| < R, |t - t_0| < R^\sigma\} \subset Q,$$

is reduced in a fraction of the cylinder $\mathcal{C}_{\gamma R}$, $\gamma < 1$, at least by a constant factor ϖ_* . This implies σ -Hölder continuity,

$$|u(x, t) - u(x_0, t_0)| \leq C (|x - x_0|^\alpha + |t - t_0|^{\alpha/\sigma}),$$

with an exponent $\alpha = \log \varpi_* / \log \gamma$.

The control of the oscillation in our nonlocal setting follows the procedure developed in [11] for a linear problem with a rough kernel (which has applications to certain nonlinear problems), combined with some ideas to deal with the nonlinearity borrowed from [3]. The operator \mathcal{L} in this latter paper is the fractional Laplacian. This allows to use Caffarelli-Silvestre's extension [13] to transform the problem into a local one. This can not be done for the general kernels that we are considering here. In fact, in the particular case of the fractional Laplacian additional regularity has been obtained [25, 45].

In the linear nonlocal setting, besides [11], which uses De Giorgi's technique, we point out the paper [28], where, by means of a different approach based on Moser's work [36, 37], the authors obtain Hölder regularity with constants that do not depend on the order of differentiability $\sigma \in (0, 2)$. See also [33] for the corresponding elliptic case. It would be interesting to see whether their method can be adapted to our problem to get rid of the σ dependence of the constants. One can find in the literature other papers dealing with heat kernel estimates and regularity issues for linear parabolic nonlocal problems with rough kernels, in the framework of Markov jump processes; see for example [4, 5, 19, 34] and the references therein.

As for nonlinear nonlocal problems with rough kernels, let us mention [18, 40], where fully nonlinear nondegenerate parabolic integro-differential equations are considered. Concerning regularity for nonlinear nonlocal equations of porous medium type, besides [3] we have [15], where, using an approach based on [11], the authors prove Hölder regularity for solutions of the so called *porous medium equation with fractional potential pressure*,

$$(1.2) \quad \partial_t u = \nabla \cdot (u \nabla p), \quad p = (-\Delta)^{-\gamma/2} u, \quad \gamma \in (0, 2).$$

For regularity results for the local filtration analogue to problem (P) we refer to [26]; see [12] for the case of powers.

After this paper was completed, we learned that, at the very same time and independently of us, Bonforte, Figalli and Ros-Oton proved in [9] the Hölder regularity of nonnegative solutions to the Cauchy-Dirichlet problem for the fractional porous medium equation $\partial_t u + (-\Delta)^{\sigma/2} u^m$, $m > 1$, in a bounded domain. The authors also indicate how the result could be extended to general unbounded domains in \mathbb{R}^N for equations of the form (P). The method of proof in that paper is completely different from ours and does not apply to solutions with sign changes.

The assumption “ u bounded” in the previous regularity result is not a big restriction as we see now. Indeed, if the kernel J satisfies the stronger assumption

$$(H'_J) \quad \frac{1}{\Lambda |x - y|^{N+\sigma}} \leq J(x, y) \leq \frac{\Lambda}{|x - y|^{N+\sigma}}, \quad x, y \in \mathbb{R}^N, \ x \neq y,$$

besides (H_J) , the natural energy associated to the operator \mathcal{L} is in fact equivalent to the fractional Sobolev energy $\|(-\Delta)^{\sigma/4}\|_2^2$; see Section 2. Then it is possible to get an L^1 – L^∞ smoothing effect repeating the Moser-like arguments in [45]. This allows in addition to get an existence and uniqueness result for initial values in $L^1(\mathbb{R}^N)$ by approximation.

Theorem 1.3 *Let J satisfy (H'_J) and $\varphi \in C^1(\mathbb{R} \setminus \{0\})$ be such that $\varphi'(s) \geq C|s|^{m-1}$ for some $m > \frac{(N-\sigma)_+}{N}$. If $u_0 \in L^1(\mathbb{R}^N)$, and $N > \sigma$, there exists a unique weak solution to the Cauchy problem (P) which is bounded in $\mathbb{R}^N \times (\tau, \infty)$ for all $\tau > 0$. This solution moreover satisfies*

$$(1.3) \quad \|u(\cdot, t)\|_\infty \leq C_1 t^{-\gamma} \|u_0\|_1^\delta,$$

with $\gamma = \frac{N}{N(m-1)+\sigma}$ and $\delta = \frac{\sigma\gamma}{N}$, the constant C_1 depending on m, σ, C , and N .

If $N = 1 \leq \sigma < 2$ the result is still valid if we assume in addition $\varphi'(s) \leq \tilde{C}|s|^{m-1}$.

We next turn our attention, in Section 5, to the asymptotic behaviour of the solutions when the operator \mathcal{L} behaves in some sense as $(-\Delta)^{\sigma/2}$ and the constitutive function φ behaves as a power in a neighbourhood of the origin. To be more precise, we assume that

$$(1.4) \quad J(x, y) = \tilde{J}(z), \quad z = |x - y|;$$

$$(1.5) \quad \lim_{|z| \rightarrow \infty} |z|^{N+\sigma} \tilde{J}(z) = \mu > 0;$$

$$(1.6) \quad \lim_{u \rightarrow 0} |u|^{1-m} \varphi'(u) = a > 0 \text{ for some } m \geq 1.$$

Under these conditions, we will prove that the solution behaves for large times as the solution $B = B_M$ to

$$(1.7) \quad \begin{cases} \partial_t B + \frac{a\mu}{m\mu_{N,\sigma}} (-\Delta)^{\sigma/2} (|B|^{m-1} B) = 0 & \text{in } Q, \\ B(\cdot, 0) = M\delta & \text{in } \mathbb{R}^N, \end{cases}$$

where δ is the unit Dirac mass placed at the origin. The constant $\mu_{N,\sigma}$, which is explicit, appears as a normalization constant in the definition of $(-\Delta)^{\sigma/2}$. Since the mass is preserved in the evolution, see Theorem 1.1, then necessarily $M = \int_{\mathbb{R}^N} u_0$. The equation in (1.7) has been analysed in [23, 24] when the initial data are integrable, and in [44] when they are non-negative Radon measures. The function B_M , obtained in the latter reference, is called a fundamental, or Barenblatt, solution; see [8] for the linear case $m = 1$. It has a definite sign, given by M , and a self-similar structure,

$$(1.8) \quad B_M(x, t) = t^{-\alpha} Z(xt^{-\beta}), \quad \alpha = \frac{N}{N(m-1) + \sigma}, \quad \beta = \frac{1}{N(m-1) + \sigma}.$$

Compactness will follow from the Hölder estimates provided by Theorem 1.2, which indeed hold, thanks to (1.6).

Theorem 1.4 *Let J and φ satisfy respectively (H_J) and (H_φ) and also (1.4)–(1.6). Let u be a bounded solution to (P), where $\varphi \in C^{1,\gamma}(\mathbb{R})$ for some $\gamma \in (0, 1)$, and $M = \int_{\mathbb{R}^N} u_0$. Then*

$$(1.9) \quad t^{\frac{N}{N(m-1) + \sigma}} \|u(\cdot, t) - B_M(\cdot, t)\|_\infty \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

The proof of (1.9) uses the existence of a solution with certain properties for a linear (dual) problem with coefficients in nondivergence form, which corresponds to a nonsymmetric kernel. This problem, which has independent interest, will be studied in the Appendix.

The corresponding result for the (local) case in which $\mathcal{L} = -\Delta$ was first obtained in [31]; see also [6, 17]. The only known result up to now for the non-local case is given in [44], where the author obtains the asymptotic behaviour for non-negative solutions for the problem with $\varphi(u) = u^m$, $m > \frac{(N-\sigma)_+}{N}$, and $\mathcal{L} = (-\Delta)^{\sigma/2}$. Let us also mention the work [16] on the asymptotic behaviour of solutions to the porous medium equation with fractional potential pressure (1.2). The large time behaviour is given again by a Barenblatt type solution, which was constructed in [7], and which turns out to be related to the Barenblatt solution of problem (P), as proved in [42].

Remarks. (i) Notice that the precise behaviour of \tilde{J} is only needed at infinity. This is due to the fact that mass goes to zero in compact sets. The behaviour far from infinity is only needed to obtain compactness, via Theorem 1.2.

(ii) The boundedness of the solution is not a restriction for a wide class of nonlinearities; see Theorem 1.3.

(iii) If $M = 0$, the result does not give a non-trivial asymptotic profile, but only that $u = o\left(t^{-\frac{N}{N(m-1) + \sigma}}\right)$, which is nevertheless better than what the smoothing effect gives; see (1.3). A nontrivial self-similar asymptotic behaviour is still expected; see [32] for the local case with power nonlinearities, where a limit dipole solution is obtained for $N = 1$. This case will be treated elsewhere.

2 Preliminaries

In this section we establish the notion of weak solution to problem (P) and fix the required functional framework.

As mentioned in the Introduction, expression (1.1) is only formal and may not make sense for the general kernels that we are considering here, even for smooth functions. The validity of (1.1) is guaranteed only for $\sigma < 1$ and functions f in $C^{\sigma+\varepsilon}$ that do not grow too much at infinity. However, we notice that if we assume the additional condition

$$(2.1) \quad J(x, x+y) = J(x, x-y),$$

then the operator has a pointwise expression, in terms of second differences, even for $1 \leq \sigma < 2$, for regular enough, $C^{1, \sigma-1+\varepsilon}$, functions,

$$(2.2) \quad \mathcal{L}f(x) = -\frac{1}{2} \int_{\mathbb{R}^N} (f(x+y) + f(x-y) - 2f(x)) J(x, x-y) dy.$$

In the case $J(x, y) = \tilde{J}(x-y)$, condition (2.1) follows from the symmetry of the kernel. We remark that, except in Section 5, where we assume that $J(x, y) = \tilde{J}(|x-y|)$, we will not impose (2.1). In any case, even if (2.1) holds, solutions to problem (P) need not be classical and we have to consider weak solutions, and in particular a weak definition of the operator.

In order to define the action of the operator \mathcal{L} in a weak sense we consider the bilinear form (nonlocal interaction energy)

$$\mathcal{E}(f, g) = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (f(x) - f(y))(g(x) - g(y)) J(x, y) dx dy,$$

and the quadratic form $\overline{\mathcal{E}}(f) = \mathcal{E}(f, f)$. For kernels satisfying the symmetry condition (2.1) and functions $f, g \in C_0^2(\mathbb{R}^N)$ we have

$$\langle \mathcal{L}f, g \rangle = \mathcal{E}(f, g);$$

see [39]. The bilinear form \mathcal{E} is well defined for more general kernels, not necessarily satisfying (2.1), and for functions in the space $\mathcal{H}_{\mathcal{L}}(\mathbb{R}^N)$, which is the closure of $C_0^\infty(\mathbb{R}^N)$ with the seminorm associated to the quadratic form $\overline{\mathcal{E}}$. We also define

$$\mathcal{H}_{\mathcal{L}}(\mathbb{R}^N) = \{f \in L^2(\mathbb{R}^N) : \overline{\mathcal{E}}(f) < \infty\}.$$

When $J(x, y) = |x-y|^{-N-\sigma}$ for some $0 < \sigma < 2$, the operator reduces to a multiple of the fractional Laplacian $(-\Delta)^{\sigma/2}$. It is clear then from (H_J) that the space $\mathcal{H}_{\mathcal{L}}(\mathbb{R}^N)$ coincides with the fractional Sobolev space

$$H^{\sigma/2}(\mathbb{R}^N) = \{f \in L^2(\mathbb{R}^N) : (-\Delta)^{\sigma/4} f \in L^2(\mathbb{R}^N)\}.$$

Actually, (H_J) implies

$$(2.3) \quad c_1 \overline{\mathcal{E}}(f) \leq \|(-\Delta)^{\sigma/4} f\|_2^2 \leq c_2 (\|f\|_2^2 + \overline{\mathcal{E}}(f)),$$

where c_1, c_2 depend only on N, σ, Λ . If we assume (H'_J) , we get the stronger result $\overline{\mathcal{E}}(f) \sim \|(-\Delta)^{\sigma/4} f\|_2^2$.

We recall also the inclusions $H^{\sigma/2}(\mathbb{R}^N) \subset L^{\frac{2N}{N-\sigma}}(\mathbb{R}^N)$ if $N > \sigma$, and $H^{\sigma/2}(\mathbb{R}) \subset L^q(\mathbb{R})$ for every $q \geq 2$ if $1 \leq \sigma < 2$. More precisely, we have the Hardy-Littlewood-Sobolev inequality [30, 41],

$$(2.4) \quad \|(-\Delta)^{\sigma/4} f\|_2 \geq c \|f\|_{\frac{2N}{N-\sigma}}, \quad N > \sigma,$$

and the Nash-Gagliardo-Nirenberg inequality [24],

$$(2.5) \quad \|(-\Delta)^{\sigma/4} f\|_2^2 \|f\|_p^p \geq c \|f\|_{\frac{N(p+2)}{2N-\sigma}}^{p+2}, \quad N \geq 1, \quad 0 < \sigma < 2, \quad p \geq 1.$$

These two inequalities combined with the upper estimate in (2.3) yield useful inclusions for functions in $\mathcal{H}_{\mathcal{L}}$.

When dealing with bounded domains $\Omega \subset \mathbb{R}^N$ we consider the operator acting on functions vanishing outside Ω . The corresponding Sobolev type space is $\mathcal{H}_{\mathcal{L},0}(\Omega)$, defined as the completion of $C_0^\infty(\Omega)$ with the norm given by $\overline{\mathcal{E}}^{1/2}$. Functions in this space satisfy a Poincaré inequality [1],

$$(2.6) \quad \overline{\mathcal{E}}(f) \geq c(\Omega) \|f\|_2^2.$$

We now define the concept of *weak solution* to the Cauchy problem (P), a function $u \in C([0, \infty) : L^1(\mathbb{R}^N))$ with $\varphi(u) \in L_{\text{loc}}^2((0, \infty) : \dot{\mathcal{H}}_{\mathcal{L}}(\mathbb{R}^N))$ such that

$$(2.7) \quad \int_0^\infty \int_{\mathbb{R}^N} u \partial_t \zeta - \int_0^\infty \mathcal{E}(\varphi(u), \zeta) = 0$$

for every $\zeta \in C_c^\infty(Q)$, and taking the initial datum $u(\cdot, 0) = u_0$ almost everywhere.

One of the tools needed in the following sections is a generalized Stroock-Varopoulos inequality; see [10, 43].

Proposition 2.1 *If F, G are two functions such that $F(u), G(u) \in \dot{\mathcal{H}}_{\mathcal{L}}(\mathbb{R}^N)$, then*

$$(2.8) \quad \overline{\mathcal{E}}(H(u)) \leq \mathcal{E}(F(u), G(u)),$$

if $(H')^2 \leq F'G'$.

3 Existence and uniqueness. Proof of Theorem 1.1

In order to perform the existence proof we rewrite the problem in the equivalent form

$$(P_\beta) \quad \partial_t \beta(w) + \mathcal{L}w = 0,$$

where $w = \varphi(u)$ and $\beta = \varphi^{-1}$.

We construct solutions by means of Crandall-Liggett's Theorem [21], which is based on an implicit in time discretization. Hence, we have to deal with the elliptic problem

$$(3.1) \quad \beta(v) + \mathcal{L}v = g \quad \text{in } \mathbb{R}^N,$$

with $g \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. To show existence of a weak solution for this problem we approximate the space \mathbb{R}^N by finite balls B_R . Thus, we look for a weak solution $v = v_R \in \mathcal{H}_{\mathcal{L},0}(B_R)$ to the problem,

$$(3.2) \quad \beta(v) + \mathcal{L}v = g \quad \text{in } B_R, \quad v = 0 \quad \text{in } B_R^c,$$

that is,

$$\mathcal{E}(v, \zeta) + \int_{B_R} \beta(v) \zeta - \int_{B_R} g \zeta = 0,$$

for every test function $\zeta \in \mathcal{H}_{\mathcal{L},0}(B_R)$. Existence is obtained in a standard way by minimizing the functional

$$J(v) = \frac{1}{2} \overline{\mathcal{E}}(v) + \int_{B_R} \Theta(v) - \int_{B_R} vg$$

in $\mathcal{H}_{\mathcal{L},0}(B_R)$, where $\Theta' = \beta$. This functional is coercive in $\mathcal{H}_{\mathcal{L},0}(B_R)$. Indeed, using Hölder's inequality, we have, for every $\varepsilon > 0$,

$$\left| \int_{B_R} vg \right| \leq \|v\|_{\frac{2N}{N-\sigma}} \|g\|_{\frac{2N}{N+\sigma}} \leq \varepsilon \|v\|_{\frac{2N}{N-\sigma}}^2 + \frac{1}{4\varepsilon} \|g\|_{\frac{2N}{N+\sigma}}^2.$$

Thus, Hardy-Littlewood-Sobolev inequality (2.4), together with Poincaré inequality (2.6), implies, if $N > \sigma$,

$$J(v) \geq C_1 \overline{\mathcal{E}}(v) - C_2.$$

For $N = 1 \leq \sigma < 2$ we use the Nash-Gagliardo-Nirenberg inequality (2.5) instead.

We have thus obtained a weak solution v_R to (3.2). On the other hand, given two data g_1 and g_2 , the corresponding weak solutions satisfy the T -contraction property

$$\int_{B_R} (\beta(v_{R,1}) - \beta(v_{R,2}))_+ \leq \int_{B_R} (g_1 - g_2)_+.$$

In particular, $\|\beta(v_R)\|_{L^1(B_R)} \leq \|g\|_{L^1(B_R)}$ and $\|\beta(v_R)\|_{L^\infty(B_R)} \leq \|g\|_{L^\infty(B_R)}$. It is then easy to prove that the monotone limit $v = \lim_{R \rightarrow \infty} v_R$ is a weak solution to problem (3.1). The T -contractivity property also holds in the limit. Moreover, $\|\beta(v)\|_{L^\infty(\mathbb{R}^N)} \leq \|g\|_{L^\infty(\mathbb{R}^N)}$ and $\|\beta(v)\|_{L^1(\mathbb{R}^N)} \leq \|g\|_{L^1(\mathbb{R}^N)}$.

Now, using Crandall-Liggett's Theorem we obtain the existence of a unique mild solution w to the evolution problem (P_β) . It is moreover a weak solution since it lies in the energy space. This is checked using the same technique as in [23], which yields, taking $\Phi' = \varphi$,

$$\int_0^T \overline{\mathcal{E}}(w) dt \leq \int_{\mathbb{R}^N} \Phi(u_0) \leq \|u_0\|_1 \|\varphi(u_0)\|_\infty \quad \text{for every } T > 0.$$

Uniqueness follows by the standard argument due to Oleinik et al. [38]; see [45]. The parabolic T -contraction can be deduced from its elliptic counterpart.

In order to complete the proof we show now that the conservation of mass is true if $|\varphi(u)| \leq C|u|^m$ above the critical exponent $\frac{(N-\sigma)_+}{N}$, and $J(x, y) = \tilde{J}(x - y)$. We adapt the technique used in the local case. Take a nonnegative non-increasing smooth cut-off function $\psi(s)$ such that $\psi(s) = 1$ for $0 \leq s \leq 1$, $\psi(s) = 0$ for $s \geq 2$, and define $\phi_R(x) = \psi(|x|/R)$. Since $\mathcal{L}\phi_R$ is well defined under our assumptions on J , we obtain, for every $t > 0$,

$$(3.3) \quad \int_{\mathbb{R}^N} u(\cdot, t) \phi_R - \int_{\mathbb{R}^N} u_0 \phi_R = - \int_0^t \mathcal{E}(\varphi(u), \phi_R) = - \int_0^t \int_{\mathbb{R}^N} \varphi(u) \mathcal{L}\phi_R.$$

On the other hand, the radial cut-off function ϕ_R has the scaling property

$$\mathcal{L}\phi_R(x) = R^{-\sigma} \tilde{\mathcal{L}}\phi_1(x/R),$$

where $\tilde{\mathcal{L}}$ is another nonlocal operator satisfying the same properties as \mathcal{L} . In particular $\tilde{\mathcal{L}}\phi_1 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. This implies $\|\mathcal{L}\phi_R\|_q \leq CR^{-\sigma+N/q}$ for every $1 \leq q \leq \infty$. Then, if we apply Hölder's inequality with $p = \max\{1, 1/m\}$ to the right-hand side of (3.3), and use the above property, together with the estimate $|\varphi(u)| \leq C|u|^m$, we get

$$\begin{aligned} \left| \int_{\mathbb{R}^N} u(\cdot, t) \phi_R - \int_{\mathbb{R}^N} u_0 \phi_R \right| &\leq t \|u_0\|_\infty^{m-1/p} \|u_0\|_1^{1/p} \|\mathcal{L}\phi_R\|_{p/(p-1)} \\ &\leq t CR^{-\sigma+N(p-1)/p} \|u_0\|_\infty^{m-1/p} \|u_0\|_1^{1/p}. \end{aligned}$$

The result follows letting R go to infinity, since the exponent of R is negative precisely for $m > \frac{N-\sigma}{N}$.

4 Regularity. Proof of Theorem 1.2

As in the previous section, it is convenient to work with equation (P_β) , so that the nonlocal term is linear, the nonlinearity being confined to the time derivative. In the course of the proof we will need to establish some estimates for the solutions of

$$(P_\vartheta) \quad \partial_t \vartheta(w) + \mathcal{K}w = 0,$$

for different functions ϑ and operators \mathcal{K} related, respectively, to our original function $\beta = \varphi^{-1}$ and our original nonlocal operator \mathcal{L} . To be more precise, ϑ will have the form $\vartheta(s) = a\beta(bs + c)$ for some $a, b > 0$. Hence, in the sequel we always assume without further mention that

$$\vartheta \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{s_0\}), \quad \vartheta'(s) > 0 \quad \text{for } s \neq s_0, \quad \text{for some } s_0 \in \mathbb{R}.$$

As for the operator \mathcal{K} , its kernel will always satisfy (H_J) .

For any given Lipschitz function ψ , we define the functional

$$\mathcal{B}_\psi(v) = \int_0^{(v-\psi)_+} \vartheta'(s+\psi)s \, ds$$

The first step of the regularity argument is to obtain, by using the equation, an energy estimate associated to \mathcal{B}_ψ . The quadratic form $\overline{\mathcal{E}}$ and the bilinear form \mathcal{E} always refer to the operator \mathcal{K} being considered.

Lemma 4.1 *Let $\psi \in C^{0,1}(\mathbb{R}^N)$ satisfy $\int_{\{|x-y|>1\}} |\psi(x) - \psi(y)|J(x,y) \, dy < C < \infty$ for every $x \in \mathbb{R}^N$, and w be a weak solution to (P_ϑ) in some finite time interval I including (t_1, t_2) . Then,*

$$(4.1) \quad \begin{aligned} \int_{\mathbb{R}^N} \mathcal{B}_\psi(w)(x, t_2) \, dx + \int_{t_1}^{t_2} \overline{\mathcal{E}}((w - \psi)_+)(t) \, dt &\leq \int_{\mathbb{R}^N} \mathcal{B}_\psi(w)(x, t_1) \, dx \\ &+ C \int_{t_1}^{t_2} \left(\int_{\mathbb{R}^N} (w - \psi)_+(x, t) + \mathbb{1}_{\{w(x,t) > \psi(x)\}} \right) \, dx \, dt. \end{aligned}$$

Proof. If we multiply equation (P_ϑ) by the function $\zeta = (w - \psi)_+$, we formally get

$$(4.2) \quad \int_{\mathbb{R}^N} \mathcal{B}_\psi(w(x, t)) \, dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \mathcal{E}(w, (w - \psi)_+)(t) \, dt = 0.$$

Though w is not differentiable in time almost everywhere, a regularization procedure in the weak formulation using some Steklov averages, following an idea from [2], allows to bypass this difficulty. In fact, it suffices to show that $\partial_t \mathcal{B}_\psi(w) \in L^2_{\text{loc}}(I : L^2(\mathbb{R}^N))$.

For any $g \in L^1(\mathbb{R}^N \times I)$ we define the Steklov average

$$g^h(x, t) = \frac{1}{h} \int_t^{t+h} g(x, s) \, ds.$$

We see that almost everywhere we have

$$\partial_t g^h(x, t) = \delta^h g(x, t) := \frac{g(x, t+h) - g(x, t)}{h}.$$

Since $\partial_t \vartheta(w)^h \in L^1(\mathbb{R}^N \times I)$, we can write the weak formulation (2.7) in the form

$$\int_0^\infty \int_{\mathbb{R}^N} \partial_t \vartheta(w)^h \zeta = \int_0^\infty \mathcal{E}(w^h, \zeta).$$

To simplify we perform the calculations with $\psi = 0$. We take $\zeta = (\chi \partial_t w^h)^{-h}$ as test function, where $\chi \in C_0^\infty(I)$, $0 \leq \chi \leq 1$, $\chi(t) = 1$ for $t \in [t_1, t_2]$, is a cut-off function. Using the “integration by parts” formulae $\int_0^\infty f \delta^h g = - \int_0^\infty g \delta^{-h} f$, and $\int_0^\infty \mathcal{E}(f, g^h) = \int_0^\infty \mathcal{E}(f^{-h}, g)$, the above identity becomes

$$\int_0^\infty \int_{\mathbb{R}^N} \chi \partial_t \vartheta(w)^h \partial_t w^h = \frac{1}{2} \int_0^\infty \chi \partial_t \mathcal{E}(w^h, w^h) = -\frac{1}{2} \int_0^\infty \partial_t \chi \mathcal{E}(w^h, w^h).$$

At this point we observe that the same calculus inequality used in [10] to prove (2.8) allows to show that $\delta^h \vartheta(w) \delta^h w \geq (\delta^h(\ell(w)))^2$, where $(\ell')^2 = \vartheta'$. We therefore get

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^N} (\delta^h(\ell(w)))^2 \leq c \int_0^\infty |\partial_t \chi'| \mathcal{E}(w^h, w^h) \leq c.$$

On the other hand, $|\delta^h B(w)| \leq |\sqrt{\vartheta'(w)} w \delta^h \ell(w)|$, so $\delta^h B(w) \in L^2(I : L^2(\mathbb{R}^N))$ provided $\sqrt{\vartheta'(w)} w \in L^\infty(\mathbb{R}^N \times I)$, and we end by passing to the limit $h \rightarrow 0$.

Once we have (4.2), the energy estimate is obtained proceeding as in [11]. \square

A consequence of this energy estimate is obtained using the properties of ϑ and \mathcal{E} . If $\ell = \inf_{\{w \geq \psi\}} w \geq 0$ and $M = \sup_{\{w \geq \psi\}} w < \infty$, we have

$$(4.3) \quad \Lambda_1(w - \psi)_+^2 \leq \mathcal{B}_\psi(w) \leq \Lambda_2(w - \psi)_+,$$

where

$$(4.4) \quad \Lambda_1 = \frac{1}{2} \inf_{\ell \leq s \leq M} \vartheta'(s), \quad \Lambda_2 = \vartheta(M) - \vartheta(\ell).$$

Therefore, using (2.3), the energy estimate (4.1) yields

$$(4.5) \quad \begin{aligned} & \Lambda_1 \int_{\mathbb{R}^N} (w - \psi)_+^2(x, t_2) dx + c \int_{t_1}^{t_2} \int_{\mathbb{R}^N} |(-\Delta)^{\sigma/4}((w - \psi)_+)(x, t)|^2 dx dt \\ & \leq \Lambda_2 \int_{\mathbb{R}^N} (w - \psi)_+(x, t_1) dx \\ & \quad + C \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \left((w - \psi)_+^2(x, t) + (w - \psi)_+(x, t) + \mathbb{1}_{\{w(x, t) > \psi(x)\}} \right) dx dt. \end{aligned}$$

This is a kind of “Anti-Sobolev inequality”, controlling the energy in terms of the size of the solution.

The next step is to obtain a first De Giorgi type oscillation reduction lemma: if w is mostly negative in space-time measure in a certain parabolic cylinder, then the supremum goes down if we restrict to a smaller nested cylinder. Due to the nonlocal character of the operator, it is necessary to have some control of the far away behaviour of the solution. This is done, as in [11], via a barrier function. In order to simplify our approach we work with normalized cylinders. The general case is treated by scaling.

Notation. $\Gamma_R = B_R(0) \times [-R^\sigma, 0]$.

Lemma 4.2 *If ϑ satisfies*

$$(4.6) \quad \delta(\vartheta) := \frac{\inf_{0 \leq s \leq 2} \vartheta'(s)}{1 + \vartheta(2) - \vartheta(0)} > 0,$$

there is a constant $c > 0$ such that if $w : \mathbb{R}^N \times (-2, 0) \rightarrow \mathbb{R}$ is a weak solution to equation (P_ϑ) satisfying

$$(4.7) \quad w(x, t) \leq 1 + (|x|^{\sigma/4} - 1)_+ \quad \text{in } \mathbb{R}^N \times (-2, 0),$$

$$(4.8) \quad |\{w > 0\} \cap \Gamma_2| \leq c\delta(\vartheta)^{1+N/\sigma},$$

then

$$w(x, t) \leq \frac{1}{2} \quad \text{if } (x, t) \in \Gamma_1.$$

Once (4.3) is true, we can perform the same proof of [11, Lemma 3.1]. Nevertheless, on the one hand we have to pursue the constants Λ_i in (4.4), to see how the nonlinearity ϑ affects the result, in the spirit of [3]. This gives the precise value of $\delta(\vartheta)$ in (4.6). On the other hand, the proof performed in [11] works only for $N > \sigma$ since Hardy-Littlewood-Sobolev inequality (2.4) is used. We complement the result for $N = 1 \leq \sigma < 2$ by using Nash-Gagliardo-Nirenberg inequality (2.5).

Proof. Let $L_k = \frac{1}{2} \left(1 - \frac{1}{2^k}\right)$. We take $\psi(x) = \psi_{L_k}(x) = L_k + (|x|^{\sigma/2} - 1)_+$ in (4.5), and put $w_k(t) = (w - \psi_{L_k})_+(\cdot, t)$. Since $\psi_k \geq 0$ we take $\ell = 0$ in (4.4). Observe that if we start the iteration from $k = 1$ we may take $\ell = 1/4$. Also, when $w > \psi_k$, condition (4.7) implies $w \leq \frac{1+\sqrt{5}}{2}$. We take $M = 2$ in (4.4) to simplify.

We define the quantity

$$U_k = \sup_{t_k < t < 0} \|w_k(t)\|_2^2 + \int_{t_k}^0 \|(-\Delta)^{\sigma/4} w_k(t)\|_2^2 dt, \quad t_k = -1 - \frac{1}{2^k}.$$

The energy estimate (4.5) implies, for $k \geq 1$, that

$$(4.9) \quad U_k \leq C2^k \frac{1 + \Lambda_2}{\Lambda_1} \int_{t_{k-1}}^{t_k} (\|w_k(t)\|_2^2 + \|w_k(t)\|_1 + \|\mathbf{1}_{\{w_k(t) > 0\}}\|_1) dt.$$

Now, since $L_k = L_{k-1} + 2^{-k-1}$, we have that $w_k > 0$ implies $w_{k-1} > 2^{-k-1}$, which in turn gives the Chebyshev type inequality

$$\int_{\mathbb{R}^N} w_k^p \leq 2^{(k+1)(q-p)} \int_{\mathbb{R}^N} w_{k-1}^q$$

for every $q > p$. Thus, for some $q > 2$ to be chosen we get that (4.9) reduces to

$$\begin{aligned} U_k &\leq C2^k \frac{1 + \Lambda_2}{\Lambda_1} \int_{t_{k-1}}^{t_k} (2^{(k+1)(q-2)} + 2^{(k+1)(q-1)} + 2^{(k+1)q}) \int_{\mathbb{R}^N} w_{k-1}^q(t) dt \\ &\leq C2^{(q+1)k} \frac{1 + \Lambda_2}{\Lambda_1} \int_{t_{k-1}}^{t_k} \|w_{k-1}(t)\|_q^q dt. \end{aligned}$$

To link this estimate with U_{k-1} , usually Hardy-Littlewood-Sobolev inequality (2.4) is used, so $N > \sigma$ is required. The following nonlinear recurrence

$$(4.10) \quad U_k \leq (C_{N,\sigma,\Lambda})^k \frac{1 + \Lambda_2}{\Lambda_1} U_{k-1}^{1+\frac{\sigma}{N}},$$

is obtained. Hence we are left with the case $\sigma \geq N$, which is only possible if $N = 1$. The idea to deal with this range of parameters is to substitute Hardy-Littlewood-Sobolev inequality by the Nash-Gagliardo-Nirenberg inequality (2.5). Using first interpolation and then (2.5) we get, with $q = 2(1 + \sigma)$,

$$\begin{aligned}
\int_{t_{k-1}}^{t_k} \|w_{k-1}(t)\|_q^q dt &\leq \int_{t_{k-1}}^{t_k} \|w_{k-1}(t)\|_2^{2(\sigma-1)} \|w_{k-1}(t)\|_{\frac{4}{2-\sigma}}^4 dt \\
&\leq \left(\sup_{t_{k-1} < t < 0} \|w_{k-1}(t)\|_2^2 \right)^\sigma \int_{t_{k-1}}^{t_k} \|(-\Delta)^{\sigma/4} w_{k-1}(t)\|_2^2 dt \\
&\leq C \left(\sup_{t_{k-1} < t < 0} \|w_{k-1}(t)\|_2^2 + \int_{t_{k-1}}^{t_k} \|(-\Delta)^{\sigma/4} w_{k-1}(t)\|_2^2 dt \right)^{1+\sigma} \\
&\leq C U_{k-1}^{1+\sigma}.
\end{aligned}$$

We get again (4.10). Thus, if $\frac{1+\Lambda_2}{\Lambda_1} U_0^{\sigma/N}$ is small, i.e.,

$$(4.11) \quad \int_{-2}^0 \int_{\mathbb{R}^N} (w - (|x|^{\sigma/2} - 1)_+)^2 < \varepsilon \left(\frac{\Lambda_1}{1 + \Lambda_2} \right)^{1+N/\sigma},$$

then $U_k \rightarrow 0$ as $k \rightarrow \infty$, which gives

$$(4.12) \quad w(x, t) \leq \frac{1}{2} + (|x|^{\sigma/2} - 1)_+ \quad \text{for } x \in \mathbb{R}^N, \quad -1 < t < 0.$$

The result now follows from a scaling argument. Let $(x_0, t_0) \in \Gamma_1$ be arbitrary, and define for some large R the function

$$w_R(x, t) = w(x_0 + R^{-1}x, t_0 + R^{-\sigma}t).$$

This function solves the equation $\partial_t \vartheta(w_R) + \mathcal{K}_R w_R = 0$, where \mathcal{K}_R is the nonlocal integral operator associated to the rescaled kernel

$$J_R(x, y) = R^{-(N+\sigma)} J(x_0 + R^{-1}x, x_0 + R^{-1}y).$$

Observe that this kernel satisfies again hypothesis (H_J) with the same constant provided $R \geq 1$. We now study the condition (4.11) for this function w_R . Observe that $1 + (|x/R|^{\sigma/4} - 1)_+ \leq |x|^{\sigma/2} - 1$ for every $|x| > R$ if R is large enough. Thus

$$\begin{aligned}
\int_{-2}^0 \int_{\mathbb{R}^N} (w_R(x, t) - (|x|^{\sigma/2} - 1)_+)^2 dx dt &\leq \int_{-2}^0 \int_{B_R(0)} (w_R)_+^2 \\
&\leq R^{N+\sigma} \int_{t_0-2/R^\sigma}^{t_0} \int_{B_1(x_0)} w_+^2 \\
&\leq R^{N+\sigma} 2^{\sigma/4} |\{w > 0\} \cap \Gamma_2|,
\end{aligned}$$

since $B_1(x_0) \times (t_0 - 2R^{-\sigma}, t_0) \subset \Gamma_2$ if $R^\sigma > 2$, and $w \leq 2^{\sigma/4}$ in $B_2(x_0)$, thanks to (4.7). Choosing $c = \varepsilon R^{-(N+\sigma)} 2^{-\sigma/4}$ in (4.8), we get from (4.12) that $w_R < 1/2$ in Γ_1 , which implies $w(x_0, t_0) < 1/2$. \square

Remark. As it is noted in the proof, the result also holds in terms of the constant

$$(4.13) \quad \bar{\delta}(\vartheta) := \frac{\inf_{1/4 \leq s \leq 2} \vartheta'(s)}{1 + \vartheta(2) - \vartheta(1/4)} > 0,$$

To proceed with the regularity proof we need to analyse what happens when the solution is neither mostly negative nor mostly positive, in the sense of Lemma 4.2, in space-time measure. To this aim we will use De Giorgi's idea of loss of mass at intermediate levels, obtaining a quantitative version of the fact that a function with a jump discontinuity cannot be in the energy space.

The key idea is to impose conditions on the nonlinearity guaranteeing that the equation is neither degenerate nor singular at the intermediate values. Hence we are in the linear setting studied in [11]. As there, the result is written in terms of the functions

$$\psi_\lambda(x) = ((|x| - \lambda^{-4/\sigma})_+^{\sigma/4} - 1)_+, \quad \lambda \in (0, 1/3),$$

used to control the growth at infinity, and

$$F(x) = \sup(-1, \inf(0, |x|^2 - 9)),$$

used to “localize” the problem in the ball B_3 . Notice that F equals -1 in B_1 , and vanishes outside B_3 .

Lemma 4.3 *Assume $C_1 \leq \vartheta'(s) \leq C_2$ for every $1/2 \leq s \leq 2$. For every $\nu, \mu > 0$ there exist $\gamma > 0$ and $\bar{\lambda} \in (0, 1/3)$ such that for any $\lambda \in (0, \bar{\lambda})$, and any solution $w : \mathbb{R}^N \times [-3, 0] \rightarrow \mathbb{R}$ to (P_ϑ) satisfying*

$$w(x, t) \leq 1 + \psi_\lambda(x) \quad \text{on } \mathbb{R}^N \times [-3, 0], \quad |\{w < 0\} \cap (B_1 \times (-3, -2))| > \mu,$$

we have the following implication: If

$$|\{w > 1 + \lambda^2 F\} \cap (B_3 \times (-2, 0))| \geq \nu,$$

then

$$|\{1 + F < w < 1 + \lambda^2 F\} \cap (B_3 \times (-3, 0))| \geq \gamma.$$

The main idea in the linear case is that the truncation function

$$\bar{\psi} = 1 + \psi_\lambda + \lambda F$$

satisfies an improved energy estimate. Since $\bar{\psi} \geq 1 - \lambda > 1/2$, and $w \leq 1 + \psi_\lambda = 1$ in B_3 , our assumptions on the nonlinearity ϑ' give

$$\frac{C_1}{2}(w - \bar{\psi})_+^2 \leq \mathcal{B}_{\bar{\psi}}(w) \leq \frac{C_2}{2}(w - \bar{\psi})_+^2,$$

and the proof in [11, Lemma 4.1] works verbatim, using the above equivalence whenever required.

We have now all the ingredients to prove the oscillation reduction result. The growth at infinity is controlled in this case by

$$H_{\lambda,\varepsilon}(x) = [(|x| - c(\lambda))^\varepsilon - 1]_+,$$

with $\lambda > 0$ small, $c(\lambda)$ large and $\varepsilon > 0$.

Lemma 4.4 *Let ϑ be such that $\delta(\vartheta) > 0$ and $\delta(\tilde{\vartheta}) > 0$, where $\tilde{\vartheta}(s) = -\vartheta(-s)$. Assume in addition that $C_1 \leq \vartheta'(s) \leq C_2$ for $s \in [1/2, 2]$ or $s \in [-2, -1/2]$. There exist constants $\varepsilon > 0$ and $\lambda^* \in (0, 1)$ such that if w is a solution to (P_ϑ) that satisfies, for $\lambda \in (0, \bar{\lambda})$ small enough,*

$$|w(x, t)| \leq 1 + H_{\lambda,\varepsilon}(x) \quad \text{for every } x \in \mathbb{R}^N, \quad -3 \leq t \leq 0,$$

then

$$\sup_{\Gamma_1} w - \inf_{\Gamma_1} w \leq 2 - \lambda^*.$$

Proof. If w (or $-w$) is subcritical at the level 0, i.e., if $|\{w > 0\} \cap \Gamma_2| \leq c\delta(\vartheta)^{1+N/\sigma}$, see (4.8), we are done thanks to Lemma 4.2. Notice that $-w$ solves (P_ϑ) with ϑ replaced by $\tilde{\vartheta}$. Otherwise, thanks to the hypotheses on ϑ' , either w or $-w$ satisfies the hypotheses of Lemma 4.3. We assume for definiteness that it is w .

We consider the sequence of rescaled functions

$$w_{k+1} = \frac{w_k - (1 - \lambda^2)}{\lambda^2}, \quad w_0 = w.$$

We have that w_k is a weak solution of problem (P_ϑ) with a nonlinearity ϑ_{k+1} given iteratively by

$$\vartheta_{k+1}(s) = \frac{1}{\lambda^2} \vartheta_k(\lambda^2 s + 1 - \lambda^2), \quad \vartheta_0 = \vartheta,$$

always with the same operator \mathcal{K} . We will prove that for each k we can apply either Lemma 4.2 or Lemma 4.3. Repeated application of Lemma 4.3 will give that in fact Lemma 4.2 can be applied after a finite number of steps. Hence we will be done.

The key point is that $\vartheta'_{k+1}(s) = \vartheta'_k(\lambda^2 s + 1 - \lambda^2)$. Hence, on the one hand, since $\lambda^2 s + 1 - \lambda^2 \in [1/2, 2]$ whenever $s \in [1/2, 2]$, we have $C_1 \leq \vartheta'_k(s) \leq C_2$ for every k . On the other hand, since $[1 - \lambda^2, 1 + \lambda^2] \subset [1/2, 2]$, we get $\delta(\vartheta_k) \geq \bar{\delta} > 0$ for all k .

Let $\nu = c\bar{\delta}^{1+N/\sigma}$, with c as in Lemma 4.2. Assume by contradiction that no w_k is subcritical, that is, $|\{w_k > 0\} \cap \Gamma_2| > \nu$ for all k , so that we could never apply Lemma 4.2. Let $\mu > 0$ be such that $|\{w < 0\} \cap (B_1 \times (-3, -2))| \geq \mu$. By construction,

$$|\{w_{k+1} < 0\} \cap (B_1 \times (-3, -2))| \geq |\{w_k < 0\} \cap (B_1 \times (-3, -2))| \geq \mu.$$

We chose λ and ε small enough, so that $\frac{H_{\varepsilon,\lambda}(x)}{\lambda^2} \leq \psi_\lambda(x)$. Since

$$w_{k+1}(x, t) \leq 1 + \frac{w_k(x, t)}{\lambda^2},$$

we get by induction that $w_k(x, t) \leq 1 + \psi_\lambda(x)$. Then, applying Lemma 4.3

$$\begin{aligned} & |\{w_{k+1} > 1 + \lambda^2 F\} \cap \Gamma_3| \\ &= |\{w_{k+1} > 1 + F\} \cap \Gamma_3| - |\{1 + \lambda^2 F > w_{k+1} > 1 + F\} \cap \Gamma_3| \\ &\leq |\{w_k > 1 + \lambda^2 F\} \cap \Gamma_3| - \gamma \leq |\{w > 1 + \lambda^2 F\} \cap \Gamma_3| - k\gamma, \end{aligned}$$

and we arrive to a contradiction if $k \geq |\Gamma_3|/\gamma$. We conclude that

$$w_{k_*} \leq \frac{1}{2} \quad \text{in } \Gamma_1 \quad \text{for some } k_* \leq |\Gamma_3|/\gamma.$$

Going back to the original variables we get that $w = 1 + \lambda^{2k_*}(w_{k_*} - 1) \leq 1 - \lambda^*$, $\lambda^* = \lambda^{2|\Gamma_3|/\gamma}/2$. \square

This result shows in particular that the oscillation of w in Γ_2 is reduced in Γ_1 by a factor $\varpi^* = 1 - \lambda^*/2$. From this we get next the regularity stated in Theorem 1.2 by means of scaling arguments. As in [14], we have to consider separately the degenerate and nondegenerate cases.

Proof of Theorem 1.2. NORMALIZATION. Let $(x_0, t_0) \in Q$ and $\tau_0 = \inf\{1, t_0/3\}$. Then

$$v_0(x, t) = \frac{w(x_0 + \tau_0^{1/\sigma} x, t_0 + \tau_0 t)}{\|w(\cdot, 0)\|_\infty}$$

is a solution to the equation

$$\partial_t \vartheta_0(v_0) + \mathcal{K}_0 v_0 = 0,$$

in $\mathbb{R}^N \times (-3, 0)$, where $\vartheta_0(s) = \frac{1}{\|w(\cdot, 0)\|_\infty} \beta(\|w(\cdot, 0)\|_\infty s)$, and the operator \mathcal{K}_0 is the nonlocal integral operator associated to the rescaled kernel

$$J_0(x, y) = \tau_0^{\frac{N+\sigma}{\sigma}} J(x_0 + \tau_0^{1/\sigma} x, x_0 + \tau_0^{1/\sigma} y).$$

The function ϑ_0 and the operator \mathcal{K}_0 satisfy the hypotheses of Lemma 4.4.

MODULUS OF CONTINUITY. We prove that v_0 is continuous at $(0, 0)$. Given $R > 1$, we define the sequence of functions, for $k \geq 1$,

$$v_k(x, t) = \frac{v_0(R^{-(k+1)}x, R^{-\sigma(k+1)}t) - \mu_k}{\varpi_k},$$

where ϖ_k and μ_k are respectively the semi-oscillation and a certain mean of v_0 in the parabolic cylinder $Q_k = \Gamma_{R^{-k}}$,

$$\varpi_k = \frac{\sup_{Q_k} v_0 - \inf_{Q_k} v_0}{2}, \quad \mu_k = \frac{\sup_{Q_k} v_0 + \inf_{Q_k} v_0}{2}.$$

They satisfy the equation

$$\partial_t \vartheta_k(v_k) + \mathcal{K}_k v_k = 0, \quad \vartheta_k(s) = \frac{\vartheta_0(\varpi_k s + \mu_k)}{\varpi_k},$$

where the operator \mathcal{K}_k has associated kernel

$$J_k(x, y) = R^{-(N+\sigma)(k+1)} J_0(R^{-(k+1)}x, R^{-(k+1)}y),$$

that satisfies again (H_J). Assuming by contradiction that $\varpi_k \geq \varsigma > 0$, we have that the function ϑ_k satisfies the hypotheses of Lemma 4.4, since $\varpi_k s + \mu_k \geq \varsigma/2$ for $s \geq 1/2$ if $\mu_k \geq 0$ and $\varpi_k s + \mu_k \leq -\varsigma/2$ for $s \leq -1/2$ if $\mu_k \leq 0$.

On the other hand, $|v_k| \leq 1 \leq 1 + H_{\lambda, \varepsilon}(x)$ for $|x| \leq R$, applying Lemma 4.4 by induction to v_{k-1} , since it can be applied to v_0 . Also, if we take $R > 1$ large enough so that $H_{\lambda, \varepsilon}(R) \geq \frac{2-\varsigma}{\varsigma}$, we get $|v_k(x, t)| \leq 1 + H_{\lambda, \varepsilon}(x)$ if $|x| \geq R$. Hence, applying Lemma 4.4 we conclude that $\varpi_k \leq (1 - \lambda^*/2)^k$, a contradiction. Therefore we have a modulus of continuity.

HÖLDER REGULARITY AT NONDEGENERACY POINTS. We assume $w(x_0, t_0) > 0$, the case $w(x_0, t_0) < 0$ being similar. We define now iteratively the sequence of functions

$$v_{k+1}(x, t) = \frac{v_k(R^{-1}x, R^{-\sigma}t) - \mu_k^*}{\varpi^*},$$

where

$$\varpi^* = 1 - \lambda^*/2, \quad \mu_k^* = \frac{\sup_{Q_1} v_k + \inf_{Q_1} v_k}{2}.$$

Observe that the recurrence relation can be written explicitly,

$$v_k(x, t) = \frac{v_0(R^{-(k+1)}x, R^{-\sigma(k+1)}t) - \nu_k}{(\varpi^*)^k},$$

where $\nu_k = \sum_{j=1}^k \mu_j^* (\varpi^*)^k$. Also, $\mu_k = (\varpi^*)^k \mu_{k+1}^* + \nu_k$, so that, since $\mu_k \rightarrow \frac{w(x_0, t_0)}{\|w(\cdot, 0)\|_\infty} > 0$, then $\nu_k \rightarrow a > 0$.

The functions v_k satisfy $|v_k| \leq 1$ in Γ_R , and the equation

$$\partial_t \vartheta_k(v_k) + \mathcal{K}_k v_k = 0,$$

where the new nonlinearity is

$$\vartheta_k(s) = \frac{\vartheta_0((\varpi^*)^k s + \nu_k)}{(\varpi^*)^k}$$

and the operator \mathcal{K}_k is as before. The function ϑ_k and the operator \mathcal{K}_k satisfy once more the hypotheses of Lemma 4.4.

On the other hand, if we take $R > 1$ large enough so that

$$H_{\lambda, \varepsilon}(x/R) \leq \frac{\varpi^*}{2} H_{\lambda, \varepsilon}(x), \quad H_{\lambda, \varepsilon}(x) \geq \frac{2(2 - \varpi^*)}{\varpi_*} \quad \text{for } |x| \geq R,$$

then $|v_k(x, t)| \leq 1 + H_{\lambda, \varepsilon}(x)$ if $|x| \geq R$. We conclude, applying Lemma 4.4, an oscillation estimate of order $(\varpi^*)^k$ for w in Q_k . This gives Hölder regularity at points where the equation is nondegenerate.

HÖLDER REGULARITY AT DEGENERACY POINTS. Let now $w(x_0, t_0) = 0$. Here we consider the sequence of functions defined by means of a recurrence that takes into account the nonlinearity, and the possible singularity of β' at zero:

$$v_{k+1}(x, t) = \frac{v_k(R^{-1}x, \gamma R^{-\sigma}t) - \mu_k^*}{\varpi^*}, \quad \gamma = \frac{\vartheta_0(\varpi^*)}{\varpi^*},$$

with μ_k^* and ϖ^* as before. The rescaled nonlinearity turns to be

$$\vartheta_k(s) = \frac{\vartheta_0((\varpi^*)^k s + \nu_k)}{\vartheta_0((\varpi^*)^k)}.$$

We observe that

$$(4.14) \quad \frac{|\nu_k|}{(\varpi^*)^k} \leq \frac{|\mu_k|}{(\varpi^*)^k} + |\mu_{k+1}^*| \leq C.$$

The conditions of Lemma 4.4 are fulfilled as long as, for every $k \geq 1$,

$$0 < C_1 \leq \frac{(\varpi^*)^k \vartheta'_0((\varpi^*)^k s + \nu_k)}{\vartheta_0((\varpi^*)^k)} \leq C_2 \quad \text{for every } s \in (1/2, 2),$$

and

$$\frac{\frac{(\varpi^*)^k}{\vartheta_0((\varpi^*)^k)} \inf_{[\nu_k, 2(\varpi^*)^k + \nu_k]} \vartheta'_0}{1 + \frac{\vartheta_0(2(\varpi^*)^k + \nu_k) - \vartheta_0(\nu_k)}{\vartheta_0((\varpi^*)^k)}} \geq \ell > 0.$$

They hold from condition (H'_φ) using (4.14). We conclude as before. \square

5 Asymptotic behaviour. Proof of Theorem 1.4

We devote this section to study the large time behaviour of bounded solutions to (P). Notice that, since the solution is bounded, we may assume (modifying the nonlinearity for large values of u , if required), that there exist constants $0 < c \leq C < \infty$ such that

$$(5.1) \quad c|u|^{m-1} \leq \varphi'(u) \leq C|u|^{m-1} \quad \text{for all } u \in \mathbb{R},$$

since this is true for $u \approx 0$. We may assume also, for simplicity, by a simple rescaling, that $a = m$ and $\mu = \mu_{N,\sigma}$ in (1.5), (1.6). On the other hand, by Theorem 1.2, we may assume that u is Hölder continuous for $t \geq 0$.

We use the nowadays classical method of scalings. Let us consider the sequence of functions

$$u_k(x, t) = k^\alpha u(k^\beta x, kt), \quad k > 0,$$

with α and β as in (1.8). Notice that the scaling preserves mass. More precisely,

$$\int_{\mathbb{R}^N} u_k(x, t) dx = \int_{\mathbb{R}^N} u(x, t) dx.$$

It is trivial to check that u_k satisfies

$$\partial_t u_k + \mathcal{L}_k \varphi_k(u_k) = 0, \quad u_k(x, 0) = k^\alpha u_0(k^\beta x),$$

where $\varphi_k(s) = k^{m\alpha} \varphi(s/k^\alpha)$, and \mathcal{L}_k has associated kernel $J_k(z) = k^{\beta(N+\sigma)} \tilde{J}(k^\beta z)$.

We first observe that the operators of the family \mathcal{L}_k and the functions of the family φ_k satisfy the hypotheses of Theorems 1.2 and 1.3. On the other hand, the assumptions (1.5) and (1.6) give

$$(5.2) \quad \lim_{k \rightarrow \infty} J_k(z) = J_\infty(z) := \mu_{N,\sigma} |z|^{-N-\sigma} \quad \text{uniformly for } |z| \geq K > 0;$$

$$(5.3) \quad \lim_{k \rightarrow \infty} \varphi_k(s) = \varphi_\infty(s) := |s|^{m-1} s \quad \text{uniformly for } |s| \leq K < \infty,$$

$$(5.4) \quad \lim_{k \rightarrow \infty} \varphi'_k(s) = \varphi'_\infty(s) \quad \text{uniformly for } |s| \leq K < \infty;$$

for some $m \geq 1$. Moreover, from the lower bound in (5.1),

$$(5.5) \quad \varphi'_k(s) \geq C|s|^{m-1} \quad \text{for some constant } C > 0 \text{ independent of } k.$$

Hence, since $\|u_{0,k}\|_{L^1(\mathbb{R}^N)} = \|u_0\|_{L^1(\mathbb{R}^N)}$, the smoothing effect (1.3) tells us that

$$(5.6) \quad \|u_k(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq c_1(\nu), \quad t \geq \nu > 0.$$

Thanks to (5.5) and (5.6), we may apply Theorem 1.2, and we obtain that the family $\{u_k\}$ is uniformly Hölder continuous for $t \geq \nu' > \nu$. Then, applying Ascoli-Arzelà's lemma, we have that there exists a subsequence converging uniformly on compact subsets of Q to some function $v \in C^\alpha(\mathbb{R}^N \times [\nu', \infty))$. Again by translation in time we may assume $\nu' = 0$. For a convergent subsequence $\{u_k\}$, since it is uniformly bounded, (5.3) yields $\varphi_k(u_k) \rightarrow |v|^{m-1} v$ uniformly in compact subsets of Q . If we are able to identify the limit v as B_M , the result will follow by the classical procedure of taking $t = 1$ and $k = t$. This identification is our next objective.

Given $\tau > 0$, we consider the translate in time of the Barenblatt solution to the fractional porous medium equation with mass $M = \int_{\mathbb{R}^N} u_0$,

$$B_{M,\tau}(x, t) = B_M(x, t + \tau).$$

Notice that $B_{M,\tau} \rightarrow B_M$ as $\tau \rightarrow 0$ weakly in $L^q(Q_T)$, $Q_T = \mathbb{R}^N \times (0, T)$, $T > 0$, $q \in (1, m + \sigma/N)$. Therefore, to show that $v = B_M$ it is enough to prove that, given any $F \in C_c^\infty(Q)$, for all $\varepsilon > 0$ there exists a value $\tau_\varepsilon > 0$, and for each τ a constant $K = K(\varepsilon, \tau)$ such that

$$\left| \iint_Q (u_k - B_{M,\tau}) F \right| \leq \varepsilon \quad \text{for } \tau \leq \tau_\varepsilon, \quad k \geq K(\varepsilon, \tau).$$

Let f be an admissible test function for our rescaled nonlinear problems. Then, for all $k > 1$ and $\tau > 0$ we have

$$\begin{aligned} & \iint_Q ((u_k - B_{M,\tau}) \partial_t f - \varphi_k(u_k) \mathcal{L}_k f + B_{M,\tau}^m (-\Delta)^{\sigma/2} f) \\ &= - \int_{\mathbb{R}^N} (u_k(\cdot, 0) - B_{M,\tau}(\cdot, 0)) f(\cdot, 0). \end{aligned}$$

This may be rewritten as

$$\begin{aligned}
& \iint_Q (u_k - B_{M,\tau}) \left(\partial_t f - \left(a_{k,\tau} + \frac{1}{n} \right) (-\Delta)^{\sigma/2} f \right) \\
&= - \underbrace{\int_{\mathbb{R}^N} (u_k(\cdot, 0) - B_{M,\tau}(\cdot, 0)) f(\cdot, 0)}_{I_1} \\
&+ \underbrace{\iint_Q (-\Delta)^{\sigma/4} (\varphi_k(B_{M,\tau}) - \varphi_\infty(B_{M,\tau})) (-\Delta)^{\sigma/4} f}_{I_2} \\
&+ \underbrace{\iint_Q \varphi_k(u_k) (\mathcal{L}_k - (-\Delta)^{\sigma/2}) f}_{I_3} + \underbrace{\frac{1}{n} \iint_Q (u_k - B_{M,\tau}) (-\Delta)^{\sigma/2} f}_{I_4},
\end{aligned}$$

where

$$a_{k,\tau} = \begin{cases} \frac{\varphi_k(u_k) - \varphi_k(B_{M,\tau})}{u_k - B_{M,\tau}} & \text{if } u_k \neq B_{M,\tau}, \\ \varphi'_k(B_{M,\tau}) & \text{if } u_k = B_{M,\tau}. \end{cases}$$

Thanks to Theorem 1.2 and the smoothness assumptions on φ , and using the upper bound in (5.1), we know that $a_{k,\tau} \in C^\alpha(Q) \cap L^\infty(Q)$.

Let now $f = f_{n,k,\tau}$ be a classical solution to

$$\partial_t f - \left(a_{k,\tau} + \frac{1}{n} \right) (-\Delta)^{\sigma/2} f = F, \quad f(x, T) = 0.$$

It is known that such a solution exists; see Appendix. Moreover, if extended by zero for $t > T$, it is an admissible test function. We now proceed to prove that $I_i \rightarrow 0$, $i = 1, \dots, 4$ for this particular choice of f . The limit is taken as $n \rightarrow \infty$, then $k \rightarrow \infty$ and finally $\tau \rightarrow 0$.

In order to estimate I_1 , we use that f is Hölder continuous and bounded (uniformly in n , k and τ ; see Theorem A.1) at $t = 0$, and that $u_k(\cdot, 0)$ and $B_{M,\tau}$ have the same integral, to obtain

$$\begin{aligned}
|I_1| &= \left| \int_{\mathbb{R}^N} (u_k(x, 0) - B_{M,\tau}(x, 0)) (f(x, 0) - f(0, 0)) dx \right| \\
&\leq C \left(R^\alpha \int_{|x| \leq R} |u_k(x, 0) - B_{M,\tau}(x, 0)| dx + \int_{|x| \geq k^\beta R} u_0(x) dx + \int_{|x| \geq R} B_{M,\tau}(x, 0) dx \right) \\
&\leq C \left(R^\alpha + \int_{|x| \geq k^\beta R} u_0(x) dx + \int_{|x| \geq R} B_{M,\tau}(x, 0) dx \right).
\end{aligned}$$

It is easily seen that we can make $|I_1| < \varepsilon$ just taking first R small enough, and then k big and τ small.

As for I_2 , we have

$$\begin{aligned}
|I_2| &\leq \|(-\Delta)^{\sigma/4} (\varphi_k(B_{M,\tau}) - \varphi_\infty(B_{M,\tau}))\|_2 \|(-\Delta)^{\sigma/4} f\|_2 \\
&\leq \sup_{0 \leq \theta \leq \|B_\tau\|_\infty} \|\varphi'_k(\theta) - \varphi'_\infty(\theta)\|_\infty \|(-\Delta)^{\sigma/4} B_{M,\tau}\|_2 \|(-\Delta)^{\sigma/4} f\|_2 \rightarrow 0
\end{aligned}$$

as $k \rightarrow \infty$ for all $\tau \in (0, \tau_\varepsilon)$ fixed, thanks to (5.4) and the uniform estimate for $\|f\|_{\dot{H}^{\sigma/2}}$ in (A.2).

The estimate for I_3 will follow from condition (5.2) and the regularity of f . In fact, since the family $\varphi_k(u_k)$ is uniformly bounded in $L^1(Q_T)$, it is enough to estimate $(\mathcal{L}_k - (-\Delta)^{\sigma/2})f$ in $L^\infty(Q_T)$. We recall that since J satisfies (1.4), we can use expression (2.2) both for \mathcal{L}_k and $(-\Delta)^{\sigma/2}$, thus getting

$$\begin{aligned} & |(\mathcal{L}_k - (-\Delta)^{\sigma/2})f(x, t)| \\ & \leq \frac{1}{2} \int_{\mathbb{R}^N} |f(x+y, t) + f(x-y, t) - 2f(x, t)| |J_k(y) - J_\infty(y)| dy \\ & \leq C \int_{|y| \leq R} \frac{|y|^{\sigma+\varepsilon}}{|y|^{N+\sigma}} dy + C \int_{|y| \geq R} |J_k(y) - J_\infty(y)| dy \rightarrow 0 \end{aligned}$$

uniformly in Q_T .

Finally using estimate (A.2) we obtain

$$\begin{aligned} |I_4| & \leq \frac{1}{n} \left(\iint_Q (u_k - B_{M,\tau})^2 \frac{1}{a_{k,\tau} + \frac{1}{n}} \right)^{1/2} \left(\iint_Q \left(a_{k,\tau} + \frac{1}{n} \right) ((-\Delta)^{\sigma/2} f)^2 \right)^{1/2} \\ & \leq C/n^{1/2}. \end{aligned}$$

Appendix. Parametrix method

We consider the nonlocal problem in non-divergence form

$$(A.1) \quad \partial_t f + a(-\Delta)^{\sigma/2} f = F \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \quad f(\cdot, 0) = f_0 \in C(\mathbb{R}^N),$$

where the coefficient $a = a(x, t)$ is Hölder continuous and satisfies the ‘ellipticity’ condition

$$0 < \lambda_1 \leq a(x, t) \leq \lambda_2 < \infty.$$

We moreover assume that $F = F(x, t)$ is also Hölder continuous. Our aim is to prove that this problem is well posed in the space

$$L_\sigma = \{g \text{ measurable} : \int_{\mathbb{R}^N} \frac{g(x)}{1 + |x|^{N+\sigma}} dx < \infty\}.$$

To this purpose we have to assume $f_0 \in L_\sigma$ and $F(\cdot, t) \in L_\sigma$ uniformly in $t \in [0, T]$.

Theorem A.1 *There is a unique classical solution $f \in C(Q_T)$ to Problem (A.1) in Q_T . It also belongs to some space $C^\alpha(\mathbb{R}^N \times (\tau, T))$ for every $\tau > 0$. Moreover, if $f_0 \in L^\infty(\mathbb{R}^N) \cap \dot{H}^{\sigma/2}(\mathbb{R}^N)$, $F \in L^\infty(Q_T) \cap L^\infty([0, T]; \dot{H}^{\sigma/2}(\mathbb{R}^N))$, then the solution*

satisfies

$$(A.2) \quad \begin{aligned} \|f\|_{L^\infty(Q_T)} &\leq C, \\ \sup_{0 < t < T} \int_{\mathbb{R}^N} |(-\Delta)^{\sigma/4} f(\cdot, t)|^2 &\leq C, \\ \sup_{0 < t < T} \int_{Q_t} a |(-\Delta)^{\sigma/2} f|^2 &\leq C, \end{aligned}$$

where the constant C depends only on $\|f_0\|_\infty$, $\|f_0\|_{\dot{H}^{\sigma/2}}$, $\|F\|_\infty$ and $\sup_{0 < t < T} \|F(\cdot, t)\|_{\dot{H}^{\sigma/2}}$.

The solution will be given by means of the representation formula

$$(A.3) \quad f(x, t) = \int_{\mathbb{R}^N} \Gamma(x, t, \xi, 0) f_0(\xi) d\xi + \int_0^t \int_{\mathbb{R}^N} \Gamma(x, t, \xi, \tau) F(\xi, \tau) d\xi d\tau,$$

where Γ is the fundamental solution to the problem. It is then clear that the regularity of f for $t > 0$ is inherited from the regularity of Γ . Even more, the regularity of Γ is determined by the regularity of the coefficient a . At this respect, assuming only local regularity, plus global boundedness, of a is enough to get local regularity of f .

Hence, our first step is to construct Γ . This will be done adapting the parametrix method of E. E. Levi [35] to the case of the nonlocal operator $L = \partial_t + a(-\Delta)^{\sigma/2}$. The fundamental solution has already been constructed in a very recent paper by Chen and Zhang [20] that has just come to our knowledge. Nevertheless, we have decided to keep our proof since it is simpler, thanks to the use of a certain quasimetric adapted to the problem. In addition, it shows clearly the local character of the regularity result, which is in fact needed in the application to the large time behaviour of solutions to (P) given in Section 5.

Theorem A.2 *There exists a function $\Gamma \in C(\mathbb{R}^{2N} \times \{\varepsilon \leq t - \tau \leq t_0\})$ satisfying $\partial_t \Gamma(\cdot, \cdot, \xi, \tau)$, $(-\Delta)^{\sigma/2} \Gamma(\cdot, \cdot, \xi, \tau) \in C^\beta(\mathbb{R}^N \times (\tau + \varepsilon, T))$ for every $\varepsilon > 0$, $T > 0$ and some $\beta \in (0, 1)$, for every fixed $\xi \in \mathbb{R}^N$, $\tau > 0$, and solving*

$$\begin{cases} \partial_t \Gamma + a(-\Delta)^{\sigma/2} \Gamma = 0, & x \in \mathbb{R}^N, \tau < t < T, \\ \Gamma(x, \tau, \xi, \tau) = \delta(x - \xi), & x \in \mathbb{R}^N. \end{cases}$$

Proof. We construct the fundamental solution of the operator $L = \partial_t + a(-\Delta)^{\sigma/2}$ in terms of the fundamental solution of the operator with frozen coefficient $L_0 = \partial_t + \bar{a}(-\Delta)^{\sigma/2}$, $\bar{a} = a(\xi, \tau)$. For that purpose let $P(x, t)$ be the Poisson kernel, which solves the fractional heat equation

$$\partial_t P + (-\Delta)^{\sigma/2} P = 0, \quad x \in \mathbb{R}^N, t > 0,$$

with $\delta(x)$ as initial value, and $P(x, t) = 0$ for $t < 0$, and define the function

$$(A.4) \quad Z(x, t, \xi, \tau) = P(x - \xi, a(\xi, \tau)(t - \tau)).$$

Then $Z(\cdot, \cdot, \xi, \tau)$ solves the fractional heat equation with constant coefficient

$$L_0 Z = \partial_t Z + \bar{a}(-\Delta)^{\sigma/2} Z = 0, \quad x \in \mathbb{R}^N, \quad t > 0.$$

This function Z , called the *parametrix*, will be the principal part of the desired function Γ . We look for Γ in the form

$$(A.5) \quad \Gamma(x, t, \xi, \tau) = Z(x, t, \xi, \tau) + \int_{\tau}^t \int_{\mathbb{R}^N} Z(x, t, \lambda, \eta) \Phi(\lambda, \eta, \xi, \tau) d\lambda d\eta,$$

where we must construct Φ in order to have $L\Gamma = 0$. The initial value is, formally,

$$\Gamma(x, \tau, \xi, \tau) = Z(x, \tau, \xi, \tau) = \delta(x - \xi).$$

Let now $\psi_0 = -LZ = (\bar{a} - a)(-\Delta)^{\sigma/2} Z$. If Φ is a fixed point of the functional

$$\mathcal{T}(\psi)(x, t, \xi, \tau) = \psi_0(x, t, \xi, \tau) + \int_{\tau}^t \int_{\mathbb{R}^N} \psi_0(x, t, \lambda, \eta) \psi(\lambda, \eta, \xi, \tau) d\lambda d\eta,$$

then the formal application of the operator L to Γ gives

$$(A.6) \quad L\Gamma = LZ + \Phi + \iint LZ\Phi = -\psi_0 + \Phi - \iint \psi_0\Phi = \Phi - \mathcal{T}(\Phi) = 0.$$

To construct Φ we define the sequence $\{\psi_k\}$ for $k \geq 1$ by the recurrence relation

$$\psi_{k+1}(x, t, \xi, \tau) = \int_{\tau}^t \int_{\mathbb{R}^N} \psi_0(x, t, \lambda, \eta) \psi_k(\lambda, \eta, \xi, \tau) d\lambda d\eta.$$

The function $\Phi = \sum_{k=0}^{\infty} \psi_k$ is clearly a fixed point of \mathcal{T} , and thus defines Γ . This is the standard construction performed in [35] of the fundamental solution, described for instance in [29]. What remains is the justification of the calculations, i.e., the convergence of the integrals involved as well as the convergence of the sum, and this is specially delicate in our situation. We first show that the sequence $\{\psi_k\}$ is well defined and that the sum is convergent. To that purpose we will use the notation $Y = (x, t)$, $\bar{Y} = (\xi, \tau) \in Q$, as well as the quasimetric $|Y - \bar{Y}|_{\sigma}$, introduced in [45],

$$(A.7) \quad |Y|_{\sigma} := \left(|x|^2 + |t|^{2/\sigma} \right)^{1/2}.$$

We also consider the corresponding Hölder space $C_{\sigma}^{\alpha}(Q)$. In terms of the joint variables Y and \bar{Y} , the function Z in (A.4) can be written as

$$Z(Y, \bar{Y}) = P(\vartheta(a(\bar{Y}))(Y - \bar{Y})),$$

where $\vartheta : \mathbb{R} \rightarrow \mathcal{M}^{N+1}$ is given by

$$\vartheta(s) = \begin{pmatrix} I & 0 \\ 0 & s \end{pmatrix},$$

$I \in \mathcal{M}^N$ being the identity matrix. With this notation the fundamental solution is

$$(A.8) \quad \Gamma(Y, \bar{Y}) = Z(Y, \bar{Y}) + \int_Q Z(Y, Y') \Phi(Y', \bar{Y}) dY'.$$

The behaviour of the coefficient function a between two positive constants makes the coefficient matrix $\vartheta(a(\bar{Y}))$ play no role in the estimates of Z needed to study Γ . Nevertheless, writing $Z(Y, \bar{Y}) = W(Y, \bar{Y}, a(\bar{Y}))$, where $W(Y, \bar{Y}, s) = P(\vartheta(s)(Y - \bar{Y}))$, allows to estimate the behaviour when moving the variable s . This will be of use later on. Now, using (A.7) we can deduce from Proposition 2.1 in [45] the estimate

$$(A.9) \quad |(-\Delta)^{\sigma/2} Z(Y, \bar{Y})| \leq \frac{c}{|Y - \bar{Y}|_{\sigma}^{N+\sigma}}.$$

This is not enough for $(-\Delta)^{\sigma/2} Z$ to be integrable near $Y = \bar{Y}$. Nevertheless, the regularity of the coefficient $(a - \bar{a})$ solves this problem. The condition $a \in C^{\alpha}(Q_T)$ means $a \in C_{\sigma}^{\alpha'}(Q_T)$, for some $\alpha' = \alpha'(\alpha, \sigma)$, though we still use the same letter α . Thus we have

$$|\psi_0(Y, \bar{Y})| \leq \frac{c}{|Y - \bar{Y}|_{\sigma}^{N+\sigma-\alpha}}$$

for $|Y - \bar{Y}|_{\sigma}$ small. For the integrability for $|Y - \bar{Y}|_{\sigma}$ large, since the time interval is bounded, we use the estimate

$$|\psi_0(Y, \bar{Y})| \leq \frac{c}{|x - \xi|^{N+\sigma}}.$$

To estimate now ψ_k we observe that, for $|x - \xi| < 1$ we have

$$\begin{aligned} |\psi_1(Y, \bar{Y})| &\leq c_1 \int_{\tau}^t \int_{|x-\lambda|<2} |\psi_0(Y, Y')| |\psi_0(Y', \bar{Y})| dY' \\ &\quad + c_1 \int_{\tau}^t \int_{|x-\lambda|>2} |\psi_0(Y, Y')| |\psi_0(Y', \bar{Y})| dY' \\ &\leq c_1 \int_{|Y-Y'|<2} \frac{1}{|Y - Y'|_{\sigma}^{N+\sigma-\alpha}} \frac{1}{|Y' - \bar{Y}|_{\sigma}^{N+\sigma-\alpha}} dY' \\ &\quad + c_2 \int_{|x-\lambda|>2} \frac{1}{|x - \lambda|^{N+\sigma}} \frac{1}{|\lambda - \xi|^{N+\sigma}} d\lambda \\ &\leq \frac{c_1}{|Y - \bar{Y}|_{\sigma}^{N+\sigma-2\alpha}} + c_2. \end{aligned}$$

We have used Lemma 1.2 of [29]. On the other hand, for $|x - \xi| > 1$ we get

$$\begin{aligned} |\psi_1(Y, \bar{Y})| &\leq c_1 \int_{\tau}^t \int_{|x-\lambda|<1/2} |\psi_0(Y, Y')| |\psi_0(Y', \bar{Y})| dY' \\ &\quad + c_2 \int_{\tau}^t \int_{|x-\lambda|>1/2} |\psi_0(Y, Y')| |\psi_0(Y', \bar{Y})| dY' \leq \frac{c}{|x - \xi|^{N+\sigma}}. \end{aligned}$$

Therefore,

$$|\psi_k(Y, \bar{Y})| \leq \frac{c_k}{|Y - \bar{Y}|_\sigma^{N+\sigma-(k+1)\alpha}}, \quad |\psi_k(Y, \bar{Y})| \leq \frac{c}{|x - \xi|^{N+\sigma}} \quad \text{for } |x - \xi| \text{ large.}$$

This means that there is a finite k_0 such that ψ_k possesses no singularity at the origin for $k \geq k_0$. Even more, it is easy to check that for large k we have the estimate $|\psi_k| \leq \frac{c^k}{k!}$. This means that the sum $\Phi = \sum_{k=0}^{\infty} \psi_k$ is absolutely convergent in compact subsets of Q_T , it is a fixed point of the functional \mathcal{T} , and satisfies the estimates

$$(A.10) \quad |\Phi(Y, \bar{Y})| \leq \frac{c}{|Y - \bar{Y}|_\sigma^{N+\sigma-\alpha}}, \quad |\Phi(Y, \bar{Y})| \leq \frac{c}{|x - \xi|^{N+\sigma}} \quad \text{for } |x - \xi| \text{ large.}$$

Also, it is easy to see that Φ is continuous in Y uniformly in \bar{Y} provided $t - \tau \geq c > 0$. In the same way Φ is continuous in \bar{Y} uniformly in Y provided $t - \tau \geq c > 0$. To prove that it is Hölder continuous, we use the formula

$$(A.11) \quad \Phi(Y, \bar{Y}) = \psi_0(Y, \bar{Y}) + \int_{Q_T} \psi_0(Y, Y') \Phi(Y', \bar{Y}) dY'.$$

Lemma A.1 *In the above hypotheses,*

$$|\Phi(Y, \bar{Y}) - \Phi(\tilde{Y}, \bar{Y})| \leq c|Y - \tilde{Y}|_\sigma^\alpha$$

for every $Y, \tilde{Y}, \bar{Y} \in Q_T$ such that $t - \tau \geq c > 0$, $|Y - \tilde{Y}|_\sigma \leq c/2$.

Proof. The first term in (A.11) is Hölder continuous, since

$$(A.12) \quad \begin{aligned} |\psi_0(Y, \bar{Y}) - \psi_0(\tilde{Y}, \bar{Y})| &\leq |a(Y) - a(\tilde{Y})| |(-\Delta)^{\sigma/2} Z(Y, \bar{Y})| \\ &\quad + |a(\tilde{Y}) - a(\bar{Y})| |(-\Delta)^{\sigma/2} Z(Y, \bar{Y}) - (-\Delta)^{\sigma/2} Z(\tilde{Y}, \bar{Y})| \\ &\leq c \left(\frac{|Y - \tilde{Y}|_\sigma^\alpha}{|Y - \bar{Y}|_\sigma^{N+\sigma}} + \frac{|Y - \tilde{Y}|_\sigma}{|\theta - \bar{Y}|_\sigma^{N+\sigma+1}} \right), \end{aligned}$$

by using again Proposition 2.1 in [45], where θ is some intermediate point between Y and \tilde{Y} . Thus, since the condition $t - \tau \geq c$ implies $|Y - \bar{Y}|_\sigma \geq c$ and $|\theta - \bar{Y}|_\sigma \geq c/2$, we get

$$|\psi_0(Y, \bar{Y}) - \psi_0(\tilde{Y}, \bar{Y})| \leq c|Y - \tilde{Y}|^\alpha.$$

As to the second term in (A.11), we combine (A.12) with (A.10) to get the desired result. \square

Now, in order to study the second term in (A.8), called the *volume potential* of Φ , we consider the volume potential for any given function $h : Q_T \rightarrow \mathbb{R}$,

$$V(h)(Y) = \int_{Q_T} Z(Y, Y') h(Y') dY'.$$

Lemma A.2 *If $h \in C_\sigma^\beta(Q_T) \cap L_\sigma(Q_T)$ for some $\beta \in (0, 1)$, then*

$$(A.13) \quad (-\Delta)^{\sigma/2} V(h)(Y) = \int_{Q_T} (-\Delta)^{\sigma/2} Z(Y, Y') h(Y') dY',$$

$$(A.14) \quad \partial_t V(h)(Y) = h(Y) + \int_{Q_T} a(Y') (-\Delta)^{\sigma/2} Z(Y, Y') h(Y') dY',$$

and, as a consequence,

$$(A.15) \quad LV(h)(Y) = h(Y) + \int_{Q_T} LZ(Y, Y') h(Y') dY'.$$

Proof. In order to prove (A.13) we only have to show that the integral on the right-hand side is well defined. The convergence of the integral at infinity is clear, as always, by the decay of the functions and the fact that h belongs to L_σ . To see the convergence at the origin we use the representation $(-\Delta)^{\sigma/2} Z(Y, Y') = (-\Delta)^{\sigma/2} W(Y, Y', a(Y'))$ defined before. But we observe that this last function satisfies, for every $Y \in Q_T$, $0 < \varepsilon < R$, and $s \neq 0$,

$$\int_{\varepsilon < |A(s)(Y - Y')|_\sigma < R} (-\Delta)^{\sigma/2} W(Y, Y', s) dY' = \frac{1}{|s|} \int_{\varepsilon < |\eta|_\sigma < R} (-\Delta)^{\sigma/2} P(\eta) d\eta = 0,$$

by Proposition 2.2 in [45]. Thus, putting $\Omega = \{\varepsilon < |A(a(Y))(Y - Y')| < R\}$, we have

$$\begin{aligned} & \left| \int_{\Omega} (-\Delta)^{\sigma/2} Z(Y, Y') h(Y') dY' \right| \\ & \leq \int_{\Omega} |(-\Delta)^{\sigma/2} W(Y, Y', a(Y')) - (-\Delta)^{\sigma/2} W(Y, Y', a(Y))| |h(Y')| dY' \\ & \quad + \int_{\Omega} |(-\Delta)^{\sigma/2} W(Y, Y', a(Y))| |h(Y) - h(Y')| dY'. \end{aligned}$$

The first term is estimated using the Mean Value Theorem,

$$|(-\Delta)^{\sigma/2} W(Y, Y', a(Y')) - (-\Delta)^{\sigma/2} W(Y, Y', a(Y))| \leq \frac{c}{|Y - Y'|_\sigma^{N+\sigma-\alpha}}.$$

The second term is estimated easily by the regularity of h and (A.9). Define now $J(x, t, \tau) = \int_{\mathbb{R}^N} Z(x, t, \xi, \tau) f(\xi, \tau) d\xi$. It satisfies

$$L_0 J = 0 \quad \text{for every } 0 < \tau < t, \quad \lim_{\tau \rightarrow t} J(x, t, \tau) = f(x, t).$$

Then, computing

$$\partial_t V(x, t) = J(x, t, t) + \int_0^T \partial_t J(x, t, \tau) d\tau$$

we get (A.14). □

In order to finish the proof of Theorem A.2 we still have to show that $L\Gamma = 0$ and that Γ takes a Dirac delta as initial value, giving a justification to the formal calculus (A.6). First we divide the integral in (A.5) in two time intervals, $[\tau, t_1] \cup [t_1, t]$. Then we apply Lemma A.2, with $h(Y)$ equal to the function $\Phi(Y, \bar{Y})$ in (A.8), to the integral in $[t_1, t]$. Since the function Φ is Hölder continuous, (A.15) holds. On the other hand, in the interval $[\tau, t_1]$ we use that $(-\Delta)^{\sigma/2}Z$ is continuous and Φ is absolutely integrable. Finally, the fact that $\lim_{t \rightarrow \tau} \Gamma(x, t, \xi, \tau) = \delta(x - \xi)$ follows immediately by checking that the last integral in (A.8) is convergent, which is now easy using all the estimates obtained so far. \square

Proof of Theorem A.1. The function f defined through the representation formula (A.3) solves the problem, all the terms appearing in the equation are continuous, and takes the initial datum in a continuous way. Uniqueness can be proved by the well-known method of the adjoint problem

$$(A.16) \quad \begin{cases} \partial_t \Gamma^* - (-\Delta)^{\sigma/2}(a(x, t)\Gamma^*) = 0, & x \in \mathbb{R}^N, T < t < \tau, \\ \Gamma^*(x, \tau, \xi, \tau) = \delta(x - \xi), & x \in \mathbb{R}^N; \end{cases}$$

see again [29]. It is easy to see that the function $\Gamma^*(Y, \bar{Y}) = \Gamma(\bar{Y}, Y)$ is its fundamental solution.

The L^∞ -norm is estimated by the maximum principle. The other two properties are obtained by multiplying the equation by $(-\Delta)^{\sigma/2}f$ and integrating in space and time. \square

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References

- [1] Andreu, F.; Mazón, J. M.; Rossi, J. D.; Toledo, J. *A nonlocal p -Laplacian evolution equation with nonhomogeneous Dirichlet boundary conditions*. SIAM J. Math. Anal. 40 (2008/09), no. 5, 1815–1851.
- [2] Aronson, D. G.; Serrin, J. *Local behavior of solutions of quasilinear parabolic equations*. Arch. Rational Mech. Anal. 25 (1967), 81–122.
- [3] Athanasopoulos, I.; Caffarelli, L. A. *Continuity of the temperature in boundary heat control problems*. Adv. Math. 224 (2010), no. 1, 293–315.
- [4] Barlow, M. T.; Bass, R. F.; Chen, Z.-Q.; Kassmann, M. *Non-local Dirichlet forms and symmetric jump processes*. Trans. Amer. Math. Soc. 361 (2009), no. 4, 1963–1999.

- [5] Bass, R. F.; Kassmann, M.; Kumagai, T. *Symmetric jump processes: localization, heat kernels and convergence*. Ann. Inst. Henri Poincar Probab. Stat. 46 (2010), no. 1, 59–71.
- [6] Biler, P.; Dolbeault, J.; Esteban, M. J. *Intermediate asymptotics in L^1 for general nonlinear diffusion equations*. Appl. Math. Lett. 15 (2002), no. 1, 101–107.
- [7] Biler, P.; Imbert, C.; Karch, G. *The nonlocal porous medium equation: Barenblatt profiles and other weak solutions*. Arch. Ration. Mech. Anal. 215 (2015), no. 2, 497–529.
- [8] Blumenthal, R. M.; Gettoor, R. K. *Some theorems on stable processes*. Trans. Amer. Math. Soc. 95 (1960), no. 2, 263–273.
- [9] Bonforte, M.; Figalli, A.; Ros-Oton, X. *Infinite speed of propagation and regularity of solutions to the fractional porous medium equation in general domains*. Comm. Pure Appl. Math., to appear. [arXiv:1510.03758v1 \[math.AP\]](#).
- [10] Brändle, C.; de Pablo, A.; *Nonlocal heat equations: decay estimates and Nash inequalities*. Preprint, [arXiv:1312.4661v3 \[math.AP\]](#).
- [11] Caffarelli, L.; Chan, C. H.; Vasseur, A. *Regularity theory for parabolic nonlinear integral operators*. J. Amer. Math. Soc. 24 (2011), no. 3, 849–869.
- [12] Caffarelli, L. A.; Friedman, A. *Regularity of the free boundary of a gas flow in an n -dimensional porous medium*. Indiana Univ. Math. J. 29 (1980), no. 3, 361–391.
- [13] Caffarelli, L.; Silvestre, L. *An extension problem related to the fractional Laplacian*. Comm. Partial Differential Equations 32 (2007), no. 7-9, 1245–1260.
- [14] Caffarelli, L.; Soria, F.; Vázquez, J. L. *Regularity of solutions of the fractional porous medium flow*. J. Eur. Math. Soc. 15 (2013), no. 5, 1701–1746.
- [15] Caffarelli, L.; Vázquez, J. L. *Nonlinear porous medium flow with fractional potential pressure*. Arch. Ration. Mech. Anal. 202 (2011), no. 2, 537–565.
- [16] Caffarelli, L. A.; Vázquez, J. L. *Asymptotic behaviour of a porous medium equation with fractional diffusion*. Discrete Contin. Dyn. Syst. 29 (2011), no. 4, 1393–1404.
- [17] Carrillo, J. A.; Di Francesco, M.; Toscani, G. *Intermediate asymptotics beyond homogeneity and self-similarity: long time behavior for $u_t = \Delta \phi(u)$* . Arch. Ration. Mech. Anal. 180 (2006), no. 1, 127–149.
- [18] Chang-Lara, H.; Dávila, G. *Regularity for solutions of non local parabolic equations*. Calc. Var. 49 (2014), 139–172.
- [19] Chen, Z.-Q. *Symmetric jump processes and their heat kernel estimates*. Sci. China Ser. A 52 (2009), no. 7, 1423–1445.

- [20] Chen, Z.-Q.; Zhang, X. *Heat kernels and analyticity of non-symmetric jump diffusion semigroups*. Probab. Theory Relat. Fields, to appear. DOI: 10.1007/s00440-015-0631-y.
- [21] Crandall, M. G.; Liggett, T. M. *Generation of semi-groups of nonlinear transformations on general Banach spaces*. Amer. J. Math. 93 (1971), 265–298.
- [22] De Giorgi, E. *Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari*. (Italian) Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3) 3 (1957) 25–43.
- [23] de Pablo, A.; Quirós, F.; Rodríguez, A.; Vázquez, J.L. *A fractional porous medium equation*. Adv. Math. 226 (2011), no. 2, 1378–1409.
- [24] de Pablo, A.; Quirós, F.; Rodríguez, A.; Vázquez, J.L. *A general fractional porous medium equation*. Comm. Pure Appl. Math. 65 (2012), no. 9, 1242–1284.
- [25] de Pablo, A.; Quirós, F.; Rodríguez, A.; Vázquez, J.L. *Classical solutions for a logarithmic fractional diffusion equation*. J. Math. Pures Appl. (9) 101 (2014), no. 6, 901–924.
- [26] de Pablo, A.; Vázquez, J.L. *Regularity of solutions and interfaces of a generalized porous medium equation in R^N* . Ann. Mat. Pura Appl. (4) 158 (1991), 51–74.
- [27] Endal, J.; Jakobsen, E.R.; del Teso, F. *Uniqueness and properties of distributional solutions of nonlocal degenerate diffusion equations of porous medium type*. Preprint, [arXiv:1507.04659 \[math.AP\]](#).
- [28] Felsinger, M.; Kassmann, M. *Local regularity for parabolic nonlocal operators*. Comm. Partial Differential Equations 38 (2013), no. 9, 1539–1573.
- [29] Friedman, A. “Partial differential equations of parabolic type”. Prentice-Hall, Inc., Englewood Cliffs, N. J., 1964.
- [30] Hardy, G. H.; Littlewood, J. E. *Some properties of fractional integrals. I*. Math. Z. 27 (1928), no. 1, 565–606.
- [31] Kamin, S. *Similar solutions and the asymptotics of filtration equations*. Arch. Rational Mech. Anal. 60 (1975/76), no. 2, 171–183.
- [32] Kamin, S.; Vázquez, J.L. *Asymptotic behaviour of solutions of the porous medium equation with changing sign*. SIAM J. Math. Anal. 22 (1991), no. 1, 34–45.
- [33] Kassmann, M. *A priori estimates for integro-differential operators with measurable kernels*. Calc. Var. Partial Differential Equations 34 (2009), no. 1, 1–21.
- [34] Komatsu, T. *Uniform estimates for fundamental solutions associated with non-local Dirichlet forms*. Osaka J. Math. 32 (1995), no. 4, 833–860.

- [35] Levi, E.E. *Sulle equazioni lineari totalmente ellittiche alle derivate parziali*. Rend. Circ. Mat. Palermo 24 (1907), no. 1, 275–317 .
- [36] Moser, J. *A Harnack inequality for parabolic differential equations*. Comm. Pure Appl. Math. 17 (1964), 101–134.
- [37] Moser, J. *On a pointwise estimate for parabolic differential equations*. Comm. Pure Appl. Math. 24 (1971), 727–740.
- [38] Oleinik, O. A.; Kalashnikov, A. S.; Czhou, Y.-I. *The Cauchy problem and boundary problems for equations of the type of non-stationary filtration*. Izv. Akad. Nauk SSSR. Ser. Mat. 22 (1958), 667–704 (Russian).
- [39] Schilling, R. L.; Uemura, T. *On the Feller property of Dirichlet forms generated by pseudo differential operators*. Tohoku Math. J. (2) 59 (2007), no. 3, 401–422.
- [40] Serra, J. *Regularity for fully nonlinear nonlocal parabolic equations with rough kernels*. Calc. Var. Partial Differential Equations 54 (2015), no. 1, 615–629.
- [41] Sobolev, S. L. *On a theorem of functional analysis*. Transl. Amer. Math. Soc. 34(2) (1963), 39–68; translation of Mat. Sb. 4 (1938) 471–497.
- [42] Stan, D.; del Teso, F.; Vázquez, J. L. *Transformations of self-similar solutions for porous medium equations of fractional type*. Nonlinear Anal. 119 (2015), 62–73.
- [43] Varopoulos, N. T. *Hardy-Littlewood theory for semigroups*. J. Funct. Anal. 63 (1985), no. 2, 240–260.
- [44] Vázquez, J. L. *Barenblatt solutions and asymptotic behaviour for a nonlinear fractional heat equation of porous medium type*. J. Eur. Math. Soc. 16 (2014), no. 4, 769–803.
- [45] Vázquez, J. L.; de Pablo, A.; Quirós, F.; Rodríguez, A. *Classical solutions and higher regularity for nonlinear fractional diffusion equations*. J. Eur. Math. Soc., to appear. [arXiv:1311.7427](#) [math.AP].

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